Ve401 Probabilistic Methods in Engineering

Sample Final Exam Exercises

The following exercises have been compiled from past final exams of Ve401. An exam will usually consist of 25-30 Points worth of such exercises to be completed in 100 minutes. In the actual exam, necessary tables of values of distributions will be provided. A PDF file with all necessary tables has been made available on Canvas. For certain of these exercises, Mathematica will be needed - in those exams, the sue fo Mathematica was permitted.

Multiple Choice

Exercise 1.
In the following exercises, mark the boxes corresponding to true statements with a cross (☒). In each case, exactly one of the provided statements is true.

i) In linear regression, a large value of R^2 indicates that

- ☐ The regression is significant, i.e., it is not likely that the coefficients $\beta_1, \ldots, \beta_p$ all vanish.
- ☐ Our fitted model will do very well when making predictions and finding confidence intervals.
- ☐ Our model explains a large proportion of the observed variation in the measured response.
- ☐ Our model is close to the true model for $\mu_{Y|x}$.

ii) Suppose that a Fisher test of the null hypothesis

$$H_0\colon \mu \le \mu_0$$

yields a very small P-value. Which of the following statements will be true?

- ☐ It is likely that the true value of μ is much larger than μ_0.
- ☐ Data was obtained that was very unusual, if the assumption is made that H_0 is true.
- ☐ It is unlikely that H_0 is true, given the data that was obtained.
- ☐ The rejection of H_0 is unlikely to be a mistake.

iii) Suppose that you are performing regression using the model $\mu_{Y|x} = \beta_0 + \beta_1 x + \beta_2 x^2$ and that you obtain a value of R^2 close to one, based on a large sample size. Which of the following statements will be true?

- ☐ There is evidence that $\beta_0 \neq 0$.
- ☐ There is evidence that $\beta_1 \neq 0$.
- ☐ There is evidence that $\beta_2 \neq 0$.
- ☐ There is evidence that $|\beta_1| + |\beta_2| \neq 0$.

iv) Suppose that you perform a test comparing a mean μ to a null value μ_0

$$H_0\colon \mu \le \mu_0,$$

After obtaining your data and from it the sample mean $\overline{X}$, you find that $\overline{X} > \mu_0$ and calculate a P-value of 0.3%. Mark all of the following statements that you think are correct.

- ☐ There is a 99.7% chance that H_0 is true.
- ☐ There is a 0.3% chance that H_0 is true.
- ☐ If H_0 were true, there would be at most a 0.3% chance of obtaining a value of $\overline{X}$ equal or greater to the one measured.
- ☐ If H_0 were false, there would be at least a 99.7% chance of obtaining a value of $\overline{X}$ equal to the one measured or greater.

(4 Marks)

Hypothesis Tests for Location and Dispersion

Exercise 2.
A manufacturer of precision measuring instruments claims that the standard deviation in the use of an instrument is not more than 0.00002 inch. An analyst, who is unaware if the claim, uses the instrument eight times and obtains a sample standard deviation of 0.00005 inch.

Using $\alpha = 0.01$, is there evidence that the manufacturer's claim is not justified?What is the power of the test if the true standard deviation equals 0.00004 inch? What is the smallest sample size that can be used to detect a true standard deviation of 0.00004 inch or more with a probability of at least 0.95? Use $\alpha = 0.01$.

(2+1+1 Marks)

Exercise 3.
The diameters of bolts are known to have a standard deviation of 0.0001 inch. A random sample of 10 bolts yields an average diameter of 0.2546 inch.

ii) Test the hypothesis that the true mean diameter of bolts equals 0.255 inch, using $\alpha = 0.05$.

ii) What size sample would be necessary to detect a true mean bolt diameter of 0.2552 inch or more with a probability of at least 0.90, assuming $\alpha = 0.05$?

(2+2 Marks)

Exercise 4.
A company wants to test whether a new assembly line procedure increases the physical stress on its workers. It selects eleven workers to work for one day using each of the assembly line procedures. At the end of each day, their pulse frequency is measured:

Procedure 1	X	63	65	71	75	72	75	68	74	62	73	72
Procedure 2	Y	80	78	96	87	88	96	82	83	77	79	71

It is thought that the median pulse frequency is higher in Procedure 2 than in Procedure 1.

i) Formulate H_0 and H_1.

ii) Use the Wilcoxon signed rank test at the 5% level of significance to determine whether you can reject H_0.

iii) Use a paired T-test (formally; the sample size is actually to small for it to give meaningful results) at the 5% level of significance to determine whether you can reject H_0.

(2 + 2 + 2 Marks)

Exercise 5.
In a hardness test, a steel ball is pressed into the material being tested at a standard load. The diameter of the indentation is measured, which is related to the hardness. Two types of steel balls are available, and their performance is compared on 10 randomly selected specimens. The hypothesis that the two steel balls give the same expected hardness measurement is to be tested at a significance level of $\alpha = 0.05$. Each specimen is tested twice, once with each ball. The results are given below:

Specimen	1	2	3	4	5	6	7	8	9	10
Ball x	75	46	57	43	58	38	61	56	64	65
Ball y	52	41	43	47	32	49	52	44	57	60

Use each of the following methods to test the hypothesis

i) A pooled T-test (assume that the variances are unequal).

ii) A Wilcoxon signed rank test.

iii) A paired T-test.

Compare the results obtained by each of the above tests. What assumptions are necessary for the validity of each test? What is your final conclusion regarding the hypothesis?
(2+2+2+3 Marks)

Exercise 6.
A product developer is interested in reducing the drying time of a primer paint. Two formulations of the paint are tested; formulation 1 is the standard chemistry, and formulation 2 has a new drying ingredient that should reduce the drying time. From experience, it is known that the standard deviation of drying time is 8 minutes, and this inherent variability should be unaffected by the addition of the new ingredient. Ten specimens are painted with formulation 1, and another 10 specimens are painted with formulation 2; the 20 specimens are painted in random order.

i) The two sample average drying times are $\overline{x}_1 = 121$ minutes and $\overline{x}_2 = 112$ minutes. Perform a significance test to judge the effectiveness of the new ingredient. What is the P-value of the test? What conclusions can you draw about the effectiveness of the new ingredient?

ii) If the true difference in mean drying times is as much as 10 minutes, find the sample sizes required to detect this difference with probability at least 0.90, assuming the hypothesis test is conducted with $\alpha = 0.01$.

(3+3 Marks)

Exercise 7.
A polymer is manufactured in a batch chemical process. Viscosity measurements are normally made on each batch, and long experience with the process has indicated that the variability in the process is fairly stable. The chemical process uses a catalyst, Catalyst A, which is intended to be replaced by the more environmentally friendly Catalyst B if the viscosity is not markedly influenced by the change in catalyst.

Let μ_A denote the mean viscosity of the polymer using Catalyst A and μ_B be the mean viscosity using Catalyst B. It is hoped that the change in catalyst will not influence the mean viscosity, but if it turns out to do so significantly, then Catalyst A will not be replaced.

i) Roughly 95% of the time, Catalyst A will lead to a polymer viscosity in the range $\mu_A \pm 2\sigma$. If μ_B differs from μ_A by at most σ, what percentage of the polymer viscosity will at most fall outside of the range $\mu_A \pm 2\sigma$?

This percentage determined in 1. is considered acceptable and catalyst A will be replaced if Catalyst B changes the mean viscosity of the polymer by less than 1 standard deviation.

ii) Formulate an appropriate hypothesis test to decide whether Catalyst A should be replaced.

iii) Given sample sizes $n_A = n_B = 20$ for the viscosities using Catalysts A and B, respectively, find the power of the test.

Pilot data yield the following viscosities:

Catalyst A	708, 732, 731, 677, 748, 702, 696, 692, 716, 729, 697, 681, 704, 740, 710, 687, 731, 704, 702, 698
Catalyst B	761, 708, 727, 730, 737, 702, 752, 758, 718, 712, 750, 747, 723, 698, 763, 756, 707, 716, 715, 732

iv) Given the above data, is the null hypothesis rejected at $\alpha = 1\%$? Quote all relevant statistics and critical values.

v) Find a 99% confidence interval on the difference in mean batch viscosity resulting from the process change.

vi) What is your conclusion regarding the catalyst change? Comment on the results of 5. and 6. above. How likely is it that you have reached the wrong conclusion?

(2+1+2+3+1+3) Marks

Chi-Squared Goodness-of-Fit Tests

Exercise 8.
The second midterm exam of the course Ve401 in Spring 2010 had 25 marks in total. The students taking the exam obtained the following marks:

> 0, 2, 4.5, 5.5, 8, 8.5, 9, 9.5, 10, 10, 10.5, 10.5, 11, 11.5, 11.5, 12, 12, 12, 12.5, 13, 13, 13.5, 13.5, 14, 14, 14, 14.5, 14.5, 15, 15, 15, 15, 15.5, 15.5, 15.5, 16, 16, 16, 16.5, 16.5, 16.5, 16.5, 16.5, 17, 17.5, 17.5, 17.5, 17.5, 17.5, 18, 18, 18, 18.5, 18.5, 18.5, 18.5, 18.5, 18.5, 18.5, 19, 19, 19, 19, 19, 19, 19, 19, 19.5, 19.5, 20, 20, 20, 20.5, 20.5, 20.5, 21, 21, 21, 21, 21, 21, 21.5, 21.5, 21.5, 21.5, 21.5, 21.5, 21.5, 22, 22, 22, 22.5, 22.5, 22.5, 22.5, 23.5, 23.5, 23.5

Let $S = \{\text{number of marks obtained in the exam}\}$ be the sample space for the the trial "student takes the second midterm exam in Ve401" and consider the random variable $X \colon S \to \mathbb{R}$, $X(s) = 25 - s$. In other words, X gives the difference between the total marks and the marks obtained by a random student.

i) The above data can be used to obtain a random sample of size $n = 98$ from X. Note again that $X(0) = 25$, $X(2) = 23, \ldots, X(23.5) = 1.5$. Plot a stem-and-leaf diagram for the values obtained for X. The stems should have integer units, i.e., there should be 25 stems.

ii) The shape of the stem-and-leaf diagram ressembles that of a chi-squared distribution. Assuming that X follows a chi-squared distribution, find a method-of-moments estimate for the degrees of freedom (rounded to one decimal point).

iii) Use a chi-squared goodness-of-fit test to test the hypotheses

$$H_0 \colon X \text{ follows a chi-squared distribution,}$$
$$H_1 \colon X \text{ does not follow a chi-squared distribution}$$

at $\alpha = 5\%$. When dividing the positive real axis into categories (intervals), it is not necessary for each interval to have the same expected number of values falling into it. But you should still make sure that the expected numbers are large enough for the chi-squared test to be applicable. If the estimated degrees of freedom are not an integer, interpolate the chi-squared table values linearly to obtain the interval boundaries.

(2+2+4 Marks)

Exercise 9.
A study is conducted to test for independence between air quality and air temperature. These data were obtained from records on 200 randomly selected days over the last few years.

	Air Quality		
Temperature	Poor	Fair	Good
Below average	1	3	24
Average	12	28	76
Above average	12	14	30

Do these data indicate an association between these variables? Explain, based on the P-value of the test.
(3 Marks)

Linear Regression

Exercise 10.
Suppose we have the following data:

x	1.0	1.0	3.3	3.3	4.0	4.0	4.0	5.6	5.6	5.6	6.0	6.0	6.5	6.5
y	1.6	1.8	1.8	2.7	2.6	2.6	2.2	3.5	2.8	2.1	3.4	3.2	3.4	3.9

i) Perform a linear regression for y as a function of x.

ii) Test the model for lack of fit at an $\alpha = 0.05$ level of significance.

(2+3 Marks)

Exercise 11.
A chemical engineer is investigating the effect of process operating temperature on product yield. The study results in the following data:

Temperature (° C)	100	120	140	160	180
Yield (%)	45	54	66	74	85

Fit a linear regression model and find

i) a 95% confidence interval for the slope;

ii) a 95% confidence interval for the intercept;

iii) a 95% confidence interval for the mean yield at 130° C;

iv) a 95% prediction interval for the yield at 130° C.

(4 × 2 Marks)

Exercise 12.
You have applied a quadratic regression model to a data set of $n = 7$ points. Your computer algebra system tells you that $R^2 = 0.781$. Is the regression significant at the $\alpha = 5\%$ level?
(3 Marks)

Exercise 13.
Consider the following data, which result from an experiment to determine the effect of x = test time in hours at a particular temperature on y = change in oil viscosity.

x	0.50	1.00	1.50	2.00	2.50
y	-0.51	-2.09	-6.03	-9.28	-17.12

i) Fit the model $\mu_{y|x} = \beta_0 + \beta_1 x + \beta_2 x^2$ to the data.

ii) Test whether the regression is significant at a 5% level.

iii) Give a 90% prediction interval for $y \mid 0$.

iv) Use an F-test to test whether a linear model is sufficient at a 5% level of significance; give SSE_{full} and $\text{SSE}_{\text{reduced}}$.

In order to solve this exercise without the use of a computer you may use that

$$(X^T X)^{-1} = \begin{pmatrix} 4.6 & -6.6 & 2. \\ -6.6 & 10.6857 & -3.4286 \\ 2. & -3.4286 & 1.1429 \end{pmatrix}$$

where X is the model determination matrix. The entries in the matrix have been rounded; make sure you use all the given decimal places in your calculations, otherwise your results will be off.
(3+3+2+3 Marks)

Exercise 14.
Consider the following data:

x	1	2	3	4	5	6	7
y	8	17	29	34	46	42	52

Fit a model of the form $\mu_{Y|x} = \beta_0 + \beta_1 x + \beta_2 x^2$. You may use that

$$(X^T X)^{-1} = \frac{1}{7} \begin{pmatrix} 17 & -9 & 1 \\ -9 & 67/12 & -2/3 \\ 1 & -2/3 & 1/12 \end{pmatrix},$$

where X is the model determination matrix. What is the value of R^2 for this model?
(3+2 Marks)

Exercise 15.

Suppose that in the simple linear regression model $Y = \beta_0 + \beta_1 x + E$ it is known that $\beta_0 = 0$, i.e., the model to be fitted is

$$\mu_{Y|x} = \beta_1 x.$$

i) Derive a least-squares estimator $\hat{\beta}_1$ for β_1.

ii) Do you expect a confidence interval for β_1 in the model $\mu_{Y|x} = \beta_1 x$ to be larger or smaller than in the model $\mu_{Y|x} = \beta_0 + \beta_1 x$? Explain!

iii) Find the distribution of $\hat{\beta}_1$ and derive a confidence interval for β_1.

(2 + 1 + 2 Marks)

Exercise 16.

Given repeated measurements in simple linear regression, we are able to decompose the error sum of squares $\mathrm{SS_E}$ into the components due to pure error $\mathrm{SS_{E,pe}}$ and due to lack-of-fit error $\mathrm{SS_{E,lf}}$,

$$\mathrm{SS_E} = \mathrm{SS_{E,pe}} + \mathrm{SS_{E,lf}}.$$

Let Y_{ij} denote the jth observation of $Y \mid x_i$, where $j = 1, \ldots, n_i$, and define

$$\overline{Y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}.$$

Then

$$\mathrm{SS_{E,pe}} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \overline{Y}_i)^2$$

and $\mathrm{SS_{E,lf}} = \mathrm{SS_E} - \mathrm{SS_{E,pe}}$.

i) Show that

$$\mathrm{SS_{E,lf}} = \sum_{i=1}^{k} n_i (\overline{Y}_i - \widehat{Y}_i)^2,$$

where $\widehat{Y}_i = b_0 + b_i x_i$.

ii) Explain in words (no formulas!) what $\mathrm{SS_{E,pe}}$ represents, based on the above sum. You should write 1-3 sentences.

iii) Explain in words (no formulas!) what $\mathrm{SS_{E,lf}}$ represents, based on the above sum. You should write 1-3 sentences.

iv) The following data represents the plasma level of polyamine (Y) in 25 children aged $0 - 4$ years old (x):

x	Y				
0	20.12	16.1	10.21	11.24	13.35
1	8.75	9.45	13.22	12.11	10.38
2	9.25	6.87	7.21	8.44	7.55
3	6.45	4.35	5.58	7.12	8.1
4	5.15	6.12	5.7	4.25	7.98

Perform a linear regression for the model $\mu_{Y|x} = \beta_0 + \beta_1 x$ on this data and test for lack of fit. State and compare the values for $\mathrm{SS_{E,pe}}$ and $\mathrm{SS_{E,lf}}$.

v) Fit a quadratic regression model $\mu_{Y|x} = \beta_0 + \beta_1 x + \beta_2 x^2$ to the above data. Is the linear model sufficient or does the quadratic model improve the fit significantly?

vi) Calculate and compare the values of R^2 for both the linear and the quadratic model. Comment on the result.

vii) In general, if repeated measurements are available, will the maximum achievable value of R^2 be greater or smaller than for data without repeated measurements? Why?

viii) Sketch the above data together with the quadratic model. Do you see any potential issues or problems with the fitted model?

(2 + 2 + 2 + 4 + 3 + 2 + 2 + 3 Marks)