Ve401 Probabilistic Methods in Engineering

Sample Midterm Exam Exercises

The following exercises have been compiled from past first midterm exams of Ve401. A first midterm will usually consist of 25-30 Points worth of such exercises to be completed in 100 minutes. In the actual exam, necessary tables of values of distributions will be provided. A PDF file with all necessary tables has been made available on Canvas.

Multiple Choice

Exercise 1.
In the following exercises, mark the boxes corresponding to true statements with a cross (☒). In each case, exactly one of the provided statements is true.

i) Let X and Y be independent random variables. Then

- ☐ $\operatorname{Var}[X - Y] = \operatorname{Var}[X] + \operatorname{Var}[Y]$
- ☐ $\operatorname{Var}[X - Y] = \operatorname{Var}[X] - \operatorname{Var}[Y]$
- ☐ $\operatorname{Var}[X - Y]^2 = \operatorname{Var}[X]^2 + \operatorname{Var}[Y]^2$
- ☐ $\operatorname{Var}[X - Y]^2 = \operatorname{Var}[X]^2 - \operatorname{Var}[Y]^2$

ii) Let X and Y be two random variables. Which of the following properties guarantees the independence of X and Y?

- ☐ $\varrho_{XY} = 0$.
- ☐ $\operatorname{E}[XY] = \operatorname{E}[X]\operatorname{E}[Y]$.
- ☐ $f_{XY}(x, y) = f_X(x) f_Y(y)$.
- ☐ $P[Y = y \mid X = x] = P[Y = y]$ and $P[X = x \mid Y = y] = P[X = x]$.

iii) An unfair coin has probability $p \neq 1/2$ for returning “heads” when tossed. It is tossed twice. Compared with a fair coin ($p = 1/2$), the probability of obtaining “two heads or two tails”

- ☐ is the same.
- ☐ is higher.
- ☐ is lower.
- ☐ can be higher or lower, depending on p.

iv) Which of the following is a property of the exponential distribution?

- ☐ The sample variance and sample mean are independent.
- ☐ The sum of two exponential random variables follows an exponential distribution.
- ☐ The mode is zero.
- ☐ The mean and the median have the same value.

v) Suppose that X follows a normal distribution. Which of the following random variables does **not** follow a normal distribution?

- ☐ $Y = X + 1$.
- ☐ $Y = 2X$.
- ☐ $Y = X^2$.
- ☐ $Y = -X$.

(5 Marks)

Elementary Probability

Exercise 2.
A company produces widgets in three factories, A, B and C. Factory A produces 20% of the widgets, factory B produces 45% of the widgets and factory C produces the remaining 35%. Of all widgets produced, 5% fail tolerance. Of those that fail tolerance, 25% were produced in factory A, 35% were produced in factory B and 40% were produced in factory C. In each factory, what percentage of the widgets produced fails tolerance?
(3 Marks)

Exercise 3.
A fair coin is tossed repeatedly.

i) What is the probability that the 5^{th} head occurs on the 10^{th} toss?

ii) Suppose that we know that the 10^{th} head occurs on the 25^{th} toss. Find the probability density function of the toss number of the 5^{th} head.

(4 Marks)

Exercise 4.
A certain widget uses a random number generator that generates random digits 0-9. However, it is known that in 20% of the widgets, the random number generator is defective and does not generate any 0s. However, the remaining digits 1-9 are generated with equal probability in the defective widgets.

A test run of the RNG in a given widget comprising twenty random numbers failed to yield the digit 0. What is the probability that this widget is one of the defective RNGs?
(5 Marks)

Exercise 5.
A certain widget has a mean time between failures of 24 hours, i.e., failures occur at a constant rate of one failure every 24 hours.

One evening, the widget was observed to be working at 10 pm and then left unobserved for the night. The next morning at 6 am, it was observed to have failed earlier. What is the probability that it was still working at 5 am that morning?
(5 Marks)

Approximating Distributions

Exercise 6.
A coin is tossed 28800 times. Assuming that the coin is fair (i.e., heads and tails each have a 50% chance of coming up), use the normal approximation to the binomial distribution to give the probability that between 14380 and 14399 heads come up.
(4 Marks)

Exercise 7.
Suppose that a hotel has 200 equally-sized rooms. From experience, it is known that 10% of travellers who have reserved a room do not turn up. Use the normal approximation to the binomial distribution to answer the following questions.

i) Suppose that the hotel has received reservations for 215 rooms. What is the probability that there will enough rooms for all travelers who turn up?

ii) Suppose that the hotel has received reservations for 215 rooms. What is the probability that more than 190 rooms (but of course not more than 200) will be occupied?

iii) At what point should the hotel stop accepting reservations if management wants to be at least 99% certain to have enough rooms for all travelers who show up?

(2+2+2 Marks)

Exercise 8.
A factory produces drill bits. It is known that 2% of the drill bits are defective. The drill bits are shipped in packages of 100. Use the Poisson approximation to the binomial distribution to answer the following questions.

i) What is the probability that a package contains no defective drill bits?

ii) What is the probability that there are no more than three defective drill bits in a package?

iii) How many drill bits must a package contain so that with probability greater than 0.9 there are at least 100 non-defective drill bits in the package?

(6 Marks)

Individual Random Variables

Exercise 9.
Let X be a discrete random variable following a Bernoulli distribution with parameter $p = 1/2$ and let $X_1, \dots, X_{10}$ be a random sample of size 10. Calculate the probability that the sample mean is greater than $3/4$, i.e., find

$$P[\overline{X} > 3/4].$$

(4 Marks)

Exercise 10.
Let X be a discrete random variable following a geometric distribution with parameter $p = 1/2$ and let $X_1, \dots, X_{10}$ be a random sample of size 10. Calculate the probability that the sample mean is no more than than 1.5, i.e., find

$$P[\overline{X} \leq 1.5].$$

(4 Marks)

Exercise 11.
The target value for the thickness of a machined cylinder is 8 cm. The upper specification limit is 8.2 cm and the lower specification limit is 7.9 cm. A machine produces cylinders that have a mean thickness of 8.1 cm and a standard deviation of 0.1 cm. Assume that the thickness of the cylinders produced by this machine follows a normal distribution.

i) What is the probability that the thickness of a randomly selected cylinder is within specification?

ii) What is the probability that the thickness of a randomly selected cylinder exceeds the target value?

iii) Find a bound $L > 0$ such that the thickness of 90% of all cylinders lies within $(8.1 \pm L)$ cm.

(2+2+2 Marks)

Exercise 12.
Consider the continuous random variable X with density

$$f_X(x) = \frac{c}{e^{-x} + e^{x}} \qquad \text{for } x \in \mathbb{R}.$$

i) Determine the constant $c \in \mathbb{R}$.

ii) Find $P[X \leq 1]$.

iii) Find the density of the random variable X^2.

(1+1+2 Marks)

Exercise 13.
The distribution function of the speed (modulus of the velocity) V of a gas molecule is described by the Maxwell-Boltzmann law

$$f_V(v) = \begin{cases} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{kT}\right)^{3/2} v^2 e^{-\frac{m}{kT} v^2/2} & v > 0 \\ 0 & v \le 0 \end{cases}$$

where $m > 0$ is the mass of the molecule, $T > 0$ is its temperature and $k > 0$ is the Boltzmann constant.

i) Find the mean and variance of V.

ii) Find the mean of the kinetic energy $E = mV^2/2$.

iii) Find the probability density f_E of E.

(3 + 2 + 2 Marks)

Multivariate Random Variables

Exercise 14.
A fair four-sided die is rolled and the result (a number from 1,. . . ,4) recorded. If the result is odd, that number is retained. But if the result is even a fair, four-sided die is rolled and that number is added to the first result. So the final result is a number between 1 and 8.

Define the two random variables

$$X = \begin{cases} 1 & \text{first die roll was even,} \\ 0 & \text{first die roll was odd,} \end{cases}$$
$$Y = \text{final result after the die roll(s).}$$

Calculate the correlation coefficient of X and Y.
(5 Marks)

Exercise 15.
Two weighted coins are available: a black coin has probability p of turning up heads, while a red coin has probability $1 - p$ of turning up heads.

On the first toss, the black coin is used. If it turns up heads, the red coin is used in the second toss. But if the black coin turns up tails, the black coin is used again in the second toss and the red coin is not used.

Let X be the number of heads on the first toss (0 or 1) and Y be the number of heads on the second toss (0 or 1).

Calculate the correlation coefficient of X and Y.
(5 Marks)

Exercise 16.
Let (X, Y) be a continuous bivariate random variable with density $f_{XY} : S \to \mathbb{R}^2$ given by

$$f_{XY}(x, y) = \begin{cases} c \cdot (x^2 + y^2) & \text{for } x^2 + y^2 \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $c \in \mathbb{R}$ is a suitable constant.

i) Determine the constant c.

ii) Find $\mathrm{E}[X]$ and $\mathrm{E}[Y]$.

iii) Find $\mathrm{Var}[X]$ and $\mathrm{Var}[Y]$.

iv) Find the correlation coefficient ϱ_{XY}.

v) Find the density of $U = X/Y$.

Helpful note: $\int_0^{2\pi} \cos^2(\theta)\, d\theta = \pi$.
(12 Marks)

Exercise 17.
Let X be a continuous random variable following a uniform distribution in the interval $[1,2]$. Let Z be an exponential random variable with random parameter $\beta = X$, i.e., Z is a random variable whose conditional density is that of an exponential distribution with parameter x.

i) Find the density function f_Z of Z.

ii) Find the expectation $\mathrm{E}[Z]$.

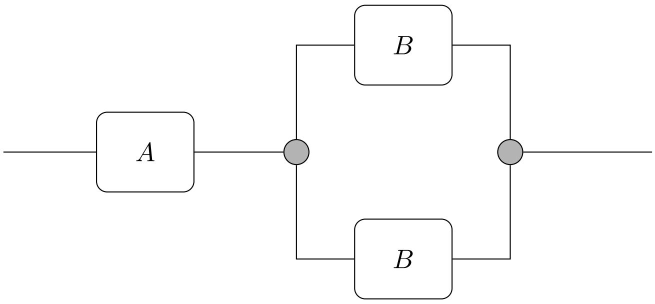
You may use the following integral identity, which holds for any $b > a > 0$:

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \ln\left(\frac{b}{a}\right).$$

(8 Marks)

Reliability

Exercise 18.
Consider the following system of components:



The system will fail if either component A or both components marked B fail. The components A and B have failure densities

$$f_A(t) = \frac{1}{100} e^{-t/100}, \qquad f_B(t) = \frac{1}{50} e^{-t/50}, \qquad t \geq 0,$$

respectively.

i) Find the reliability functions $R_A(t)$ and $R_B(t)$ of component A and each component B.

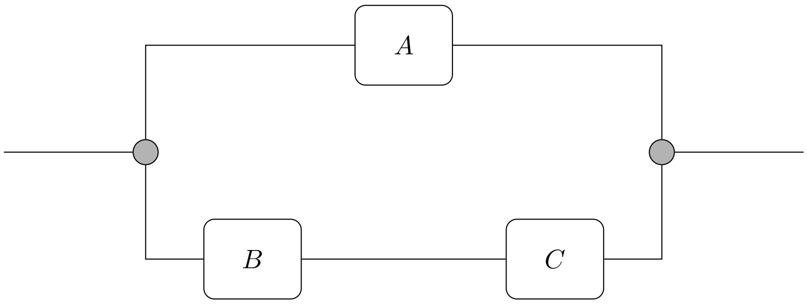
ii) Find the reliability function $R(t)$ of the system.

iii) What is the expected time of failure of each individual component?

iv) What is the expected time of failure of the system?

(2+2+2+2 Marks)

Exercise 19.
Consider the following system of components:



The components A, B and C have failure densities

$$f_A(t) = \frac{t}{50} e^{-t^2/100}, \qquad f_B(t) = \frac{1}{40} e^{-t/40}, \qquad f_C(t) = \frac{1}{25} e^{-t/25}, \qquad t \geq 0,$$

respectively.

i) Find the reliability functions $R_A(t)$, $R_B(t)$ and $R_C(t)$.

ii) Find the reliability function $R(t)$ of the system.

iii) What is the expected time of failure of each component? Evaluate all integrals explicitly; there should be no gamma functions or other integrals in your answer.

iv) What is the expected time of failure of the system? Evaluate all integrals explicitly; there should be no gamma functions or other integrals in your answer.

(2+2+2+3 Marks)

Estimators

Exercise 20.
Let X be a continuous random variable with density

$$f_\theta(x) = \begin{cases} \dfrac{\theta+1}{x^{\theta+2}} & \text{for } x > 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is a parameter. Find the method-of-moments estimator for the parameter θ.
(3 Marks)

Exercise 21.
Let (X, f_X) be a continuous random variable following a uniform distribution on the interval $[0, \theta]$, $\theta > 0$, i.e.,

$$f_X(x) = \begin{cases} 1/\theta & 0 \leq x \leq \theta \\ 0 & \text{otherwise.} \end{cases}$$

i) Find the method-of-moments estimator for θ.

ii) Find the maximum-likelihood estimator for θ.

(2+3 Marks)

Exercise 22.
A certain type of harddrive is known to have a lifetime given by an exponential distribution with (unknown) parameter $\beta > 0$. To estimate β, n identical and independent hard drives are tested and their times of failure recorded. After time $T > 0$ the test is stopped and $n - m$ hard drives are found to be still working.

Find the maximum likelihood estimator for β.
(5 Marks)

Exercise 23.
Let X be a continuous random variable with density

$$f_\theta(x) = \begin{cases} \dfrac{x}{\theta} e^{-x^2/(2\theta)} & \text{for } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is a parameter. Find the maximum likelihood estimator for θ.
(5 Marks)

Data Visualization, Interpretation and Confidence Intervals

Exercise 24.
Consider the following set of 36 data:

9.2	6.7	7.6	6.2	7.8	6.0
4.6	4.4	7.9	10.1	5.6	7.8
6.1	9.2	5.4	6.0	7.6	3.5
6.6	8.2	3.9	4.9	8.8	5.4
4.4	7.3	7.2	12.2	4.6	4.2
9.8	3.0	6.0	8.6	4.8	8.2

Find the quartiles and the interquartile range for the data. Create a histogram using the Freedman-Diaconis bin widths. Does the data appear to come from a normal distribution? Give your reasoning!
(6 Marks)

Exercise 25.
Consider the following set of 36 data:

9.2	6.7	7.6	6.2	7.8	6.0
4.6	4.4	7.9	10.1	5.6	7.8
6.1	9.2	5.4	6.0	7.6	3.5
6.6	8.2	3.9	4.9	8.8	5.4
4.4	7.3	7.2	12.2	4.6	4.2
9.8	3.0	6.0	8.6	4.8	8.2

Find the quartiles and the interquartile range for the data. Create a box-and-whisker diagram. Does the data appear to come from a normal distribution? Give your reasoning!
(6 Marks)

Exercise 26.
Consider the following set of 34 data, taken from a normal distribution:

5.8	6.0	7.8	7.4	6.2	6.1
8.6	9.7	4.4	4.4	7.6	9.0
9.2	6.4	7.0	8.7	6.0	6.1
6.8	10.	5.3	8.1	8.9	5.3
6.7	5.6	3.7	8.7	6.6	10.1
6.9	6.6	6.4	5.1		

Find a two-sided, 90% confidence interval for the variance.
(3 Marks)

Exercise 27.
Consider the following set of 34 data, taken from a normal distribution:

5.8	6.0	7.8	7.4	6.2	6.1
8.6	9.7	4.4	4.4	7.6	9.0
9.2	6.4	7.0	8.7	6.0	6.1
6.8	10.	5.3	8.1	8.9	5.3
6.7	5.6	3.7	8.7	6.6	10.1
6.9	6.6	6.4	5.1		

Find a two-sided, 95% confidence interval for the mean.
(3 Marks)