# Ve401 Probabilistic Methods in Engineering

## Summer 2021 — Assigment 1

**Date Due: 12:10 PM, Friday, the 21st of May 2021**

This assignment has a total of **(23 Marks)**.

**Exercise 1.1 Some Basic Properties of a Probability Space**
Suppose that a probability space $(S, \mathcal{F}, P)$ is given. Let $A, B \in \mathcal{F}$. Show that

i) $A \cap B \in \mathcal{F}$.

ii) If $A \subset B$, then $P[A] \leq P[B]$.

iii) If $A$ and $B$ are mutually exclusive events with non-zero probabilities, then $A$ and $B$ are not independent.

**(3 Marks)**

**Exercise 1.2 Risk**
In the board game *Risk*, a player (Player 1) with at least three armies in Territory A attacks another player (Player 2) with at least two armies in Territory B.

i) Suppose that Player 1 "attacks with three armies" and Player 2 "defends with one army". Player 1 rolls three red-colored dice and Player 2 rolls one blue-colored die. The highest result among the red dice is compared with the result of the blue die (the other red die results are ignored). If is strictly greater, Player 2 loses one army. If it is equal or less than the blue die result, Player 1 loses one army.

   Find the probability that Player 1 loses an army.
   **(2 Marks)**

ii) Suppose that Player 1 "attacks with three armies" and Player 2 "defends with two armies". Player 1 rolls three red-colored dice and Player 2 rolls two blue-colored dice. The highest result among the red dice is compared with the highest result of the blue dice and the second-highest results (which could be equal to the highest result if two dice show that number) of both dice are also compared with each other. (The remaining red die result is discarded.) In both cases, Player 2 loses an army if the red die result is strictly larger than the blue die result, otherwise Player 1 loses an army.

   Find the probabilities that

   - Player 1 loses two armies;
   - Player 2 loses two armies;
   - Both players lose one army each.

   **(3 Marks)**

**Exercise 1.3 Two Children Paradox Redux**
Suppose that it is known that Mr. Smith has two children, one of which is a girl.

i) Suppose it is known that the girl was born between 12:00:00 pm and 11:59:59 pm (you may assume that this is the case for one-half of all newborns). What is the probability that the other child is a girl?
   **(2 Marks)**

ii) More generally, if the girl can be characterized by a property that holds for $p \cdot 100\%$ of all children, what is the probability that the other child is a girl?
   **(2 Marks)**

**Exercise 1.4 Too Many Doors!**
Today's episode of the Monty hall show features $n > 3$ doors, one of which has a prize behind it. After you choose a door, Monty Hall will open $k < n - 1$ doors that do not contain a prize. You may then change your choice, selecting a random door among the remaining $n-k-1$ doors. Based on $n$ and $k$, what is your probability of winning the prize?
**(3 Marks)**

In clinical testing for a disease, one is interested in whether a patient is healthy ($h$) or diseased ($d$). A test for the disease may be either positive ($p$), indicating that a patient is diseased, or negative ($n$), indicating that the patient is healthy. A test is never completely reliable and there are four possible outcomes:

- $n \mid h$ – A healthy person tests negative for the disease.
- $p \mid h$ – A healthy person tests positive for the disease (*false positive*).
- $n \mid d$ – A diseased person tests negative for the disease (*false negative*).
- $p \mid d$ – A diseased person tests positive for the disease.

The probability $P[n \mid h]$ is called the *specificity*, $P[p \mid d]$ the *sensitivity* of the test.

**Exercise 1.5 Testing for COVID-19**

During the COVID-19 pandemic of 2020, testing for presence of the disease quickly became critical. Since the genome of the virus was available early, initial efforts focussed on detecting the viral RNA in a patient's bloodstream using a "reverse transcriptase polymerase chain reaction" (RT-PCR) procedure, commonly referred to as a "PCR test". (Subsequently, other tests were developed.)

A guideline[1] published in May 2020 assumed a specificity of 70% and a sensitivity of 95% for PCR tests, based on early reports of the reliability of PCR tests.

i) Suppose that a random person from a population where the COVID-19 incidence is 1% is tested for the disease. Find $P[d \mid p]$, $P[d \mid n]$, $P[h \mid p]$, $P[h \mid n]$. **(2 Marks)**

ii) Interpret the above numbers and explain what this says about the reliability of the test in such a situation. What practical consequences would you recommend? **(2 Marks)**

iii) Suppose that a person with typical flu symptoms presents in a clinic and is tested for COVID-19. From experience, 30% of such patients are diseased. Find $P[d \mid p]$, $P[d \mid n]$, $P[h \mid p]$, $P[h \mid n]$. **(2 Marks)**

iv) Interpret the above numbers and explain what this says about the reliability of the test in such a situation. What practical consequences would you recommend? **(2 Marks)**

[1] Watson, J., Whiting, P. F., and Brush, J. E. (2020). *Interpreting a covid-19 test result.* BMJ (Clinical research ed.), 369, m1808. `https://doi.org/10.1136/bmj.m1808`