Ve401 Probabilistic Methods in Engineering

Summer 2021 — Assigment 2

Date Due: 12:10 PM, Friday, the 28th of May 2021

This assignment has a total of **(26 Marks)**.

Exercise 2.1 Discrete Uniform Distribution

A discrete random variable is said to be *uniformly distributed* if it assumes a finite number of values with each value occurring with the same probability. For example, in the generation of a single random digit taken from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ the number generated is uniformly distributed with each possible digit occurring with probability $1/10$.

In general, the density for a uniformly distributed random variable $X\colon S \to \{x_1, \dots, x_n\} \subset \mathbb{R}$, $n \in \mathbb{N}$, is given by

$$f(x_k) = \frac{1}{n}, \qquad\qquad k = 1, \dots, n.$$

i) Find the moment-generating function for a discrete uniform random variable.
(2 Marks)

ii) Use the moment-generating function to find $\mathrm{E}[X]$ and $\mathrm{Var}[X]$.
(2 Marks)

Exercise 2.2 Rolling Two Dice

A pair of dice[1] is rolled repeatedly. Find the probability that the event "the sum of the dice is 7" occurs before the event "the sum of the dice is 5."
(3 Marks)

Exercise 2.3 An Expression for the Expectation

Show that if $X\colon S \to \Omega \subset \mathbb{N} \setminus \{0\}$ is a discrete random variable whose expectation and variance exist, then

$$\mathrm{E}[X] = \sum_{x=1}^{\infty} P[X \ge x], \qquad\qquad \mathrm{E}[X^2] = \sum_{x=1}^{\infty} (2x-1) P[X \ge x].$$

(5 Marks)

Exercise 2.4 Maximum of Independent Discrete Random Variables

Two discrete random variables X and Y are said to be independent if

$$P[X = x \text{ and } Y = y] = P[X = x] \cdot P[Y = y] \qquad\qquad \text{for any } x \in \operatorname{ran} X \text{ and } y \in \operatorname{ran} Y.$$

i) Let $Z = \max(X, Y)$, where X and Y are assumed to be independent and $X, Y\colon S \to \{1, \dots, n\}$. Show that

$$\mathrm{E}[Z] = \sum_{k=1}^{n} \big(1 - P[X < k] \cdot P[Y < k]\big)$$

(2 Marks)

ii) A fair six-sided die is rolled twice and the maximum of the outcomes is retained. What is the expected value of the retained outcome?
(2 Marks)

Exercise 2.5 Binomial Distribution, but how, exactly?

A jewel thief on the run is forced to discard 20 diamonds into a sack of flour. Later, this flour is used to bake 100 large loaves of bread. What is the probability that the first 10 loaves of bread contain three diamonds?
(4 Marks)

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[1] Adapted from U. Krengel, *Einführung in die Wahrscheinlichkeitstheorie und Statistik* [Introduction to Probability and Statistics], 5th Ed., Vieweg Verlag, 2000

Exercise 2.6 Interpreting an Experiment - From Math to Real Life

You are given a coin which is claimed to be fair.

i) If the coin is in fact fair and is tossed 100 times, give the smallest number N so that with a probability of 99% the number of "heads" lies within the interval $[50 - N, 50 + N]$. (This may involve some tedious calculations; feel free to use Mathematica.)
(2 Marks)

ii) You perform this experiment, i.e., toss the coin 100 times, and you observe a number of "heads" that lies outside the above interval. Do you conclude that the coin is not fair?

Instructions: Please start your answer with the words "I conclude. . . " or "I do not conclude. . . " and spend at least several sentences explaining your reasoning. You should use your entire knowledge of probability (e.g., Bayes's theorem) to support your argument and you are encouraged to make personal judgements (e.g., "I think that 99% is a large/small number, therefore. . . "), comment on the situation in general or otherwise answer freely. **This is an essay question.**
(4 Marks)