Ve401 Probabilistic Methods in Engineering

Summer 2021 — Assigment 3

Date Due: 12:10 PM, Friday, the 4th of June 2021

This assignment has a total of **(30 Marks)**.

Exercise 3.1 Poisson Approximation in Practice

A 400-page book contains 400 typographical errors, randomly distributed on its pages. Approximate the probability that the first two pages of the book contain exactly one error each.
(4 Marks)

Exercise 3.2 Cumulative Distribution Function of the Poisson Distribution

Let X follow a Poisson distribution with parameter $k > 0$. Show that

$$P[X \le n] = \frac{1}{n!} \int_k^\infty x^n e^{-x}\,dx.$$

(4 Marks)

Exercise 3.3 Continuous Uniform Distribution

A continuous random variable X is said to be *uniformly distributed* over an interval (a, b) if its density is given by

$$f(x) = \begin{cases} 1/(b-a) & \text{for } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

Find the moment-generating function for a uniformly distributed random variable X and show that

$$\mathrm{E}[X] = (a+b)/2 \qquad \text{and} \qquad \mathrm{Var}[X] = (b-a)^2/12.$$

(4 Marks)

Exercise 3.4 Cauchy Distribution

Let $\alpha > 0$ and $\beta \in \mathbb{R}$ be given and define the continuous random variable X via the density

$$f_{\alpha,\beta}(x) = \frac{c}{\alpha^2 + (x-\beta)^2}$$

where $c > 0$ is a constant. (X is said to follow a *Cauchy distribution*.)

i) Find the constant c.

ii) Find the expectation and variance of X, if they exist. If either does not exist, show this.

(4 Marks)

Exercise 3.5 Transforming a Single, Continuous Random Variable

Let X be a continuous random variable with density

$$f_X(x) = \begin{cases} \frac{1}{\pi^2}x & 0 \le x < \pi, \\ \frac{2}{\pi} - \frac{1}{\pi^2}x & \pi \le x \le 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Z = \cos(X)$.

i) Find the density function f_Z of Z.
(2 Marks)

ii) Find the expectation $\mathrm{E}[Z]$.
(2 Marks)

Exercise 3.6 Finding Probabilities with the Normal Distribution

The compressive strength of samples of cement can be modeled by a normal distribution with a mean of 6000 kilograms per square centimeter and a standard deviation of 100 kilograms per square centimeter.

i) What is the probability that a sample's strength is less than 6250 kg / cm^2?
(1 Mark)

ii) What is the probability that a sample's strength is between 5800 and 5900 kg / cm^2?
(1 Mark)

iii) What strength is exceeded by 95% of the samples?
(2 Marks)

Exercise 3.7 In Games, Expectation is not Everything

A person invests one dollar in an investment fund. Each year, the investment fund has a 40% chance of doubling in value and a 60% chance of halving in value. These probabilities are identical each year.

i) What is the expected value of the investment fund after one year? What is the expected value after 100 years?
(2 Marks)

ii) What is the probability of the investment fund being worth more than 1 dollar after 100 years? Write down the exact formula using the binomial distribution. Then use the normal distribution to approximate this probability.
(4 Marks)