Ve401 Probabilistic Methods in Engineering

Summer 2021 — Assigment 4

Date Due: 12:10 PM, Friday, the 11$^{\text{th}}$ of June 2021

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This assignment has a total of **(35 Marks)**.

Exercise 4.1 Correlation

Suppose that two independent and identical coins are flipped, with results 1 (heads) or 0 (tails). Let $P[1] = p$, $P[0] = 1 - p$ for some $0 < p < 1$. The sample space is given by

$$S = \{(0,0), (0,1), (1,0), (1,1)\}.$$

Define the two random variables

$$X(i,j) = \min\{i,j\}, \qquad\qquad Y(i,j) = i - j.$$

Calculate the correlation coefficient of X and Y.
(4 Marks)

Exercise 4.2 Bivariate Normal Distribution as a Mixture of Independent Normal Distributions

Suppose that X_1 and X_2 follow independent normal distributions with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively.

i) Let $\boldsymbol{X} = (X_1, X_2)$. Show that the joint density of $\boldsymbol{X}$ is given by

$$f_{\boldsymbol{X}}(x) = f_{\boldsymbol{X}}(x_1, x_2) = \frac{1}{2\pi\sqrt{\det \varSigma_{\boldsymbol{X}}}} e^{-\frac{1}{2}\langle x - \mu_{\boldsymbol{X}}, \varSigma_X^{-1}(x - \mu_{\boldsymbol{X}})\rangle}$$

where $\mu_{\boldsymbol{X}} = (\mu_1, \mu_2)$ and $\varSigma_{\boldsymbol{X}} = \operatorname{diag}(\sigma_1^2, \sigma_2^2)$ is the 2×2 matrix with the variances on the diagonal and all other entries vanishing.

ii) Let A be an invertible 2×2 matrix and define $\boldsymbol{Y} = A\boldsymbol{X}$. Show that the joint density of $\boldsymbol{Y}$ is given by

$$f_{\boldsymbol{Y}}(y) = \frac{1}{2\pi\sqrt{|\det \varSigma_{\boldsymbol{Y}}|}} e^{-\frac{1}{2}\langle y - \mu_{\boldsymbol{Y}}, \varSigma_{\boldsymbol{Y}}^{-1}(y - \mu_{\boldsymbol{Y}})\rangle} \qquad (*)$$

where $\mu_{\boldsymbol{Y}} = \operatorname{E}[\boldsymbol{Y}]$, $\varSigma_{\boldsymbol{Y}} = \operatorname{Var}[\boldsymbol{Y}]$ and $\langle \cdot\,,\cdot \rangle$ denotes the euclidean scalar product in $\mathbb{R}^2$.

iii) Show that $(*)$ may be expressed as

$$f_{\boldsymbol{Y}}(y_1, y_2) = \frac{1}{2\pi\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\varrho^2}} e^{-\frac{1}{2(1-\varrho^2)}\left[\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)^2 - 2\varrho\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)\left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right) + \left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right)^2\right]} \qquad (**)$$

where μ_{Y_i} is the mean and $\sigma_{Y_i}^2$ the variance of Y_i, $i = 1, 2$, and ϱ is the correlation of Y_1 and Y_2.
(2 Marks)

Exercise 4.3 Drawing until First Success in the Hypergeometric Setting

A box contains N balls, of which r are red and $N - r$ are black. Balls are drawn from the box until a red ball is drawn. Show that the expected number of draws is

$$\frac{N+1}{r+1}.$$

Hint: You may use that $\sum_{x=0}^{r} \binom{N-r+x}{N-r} = \binom{N+1}{N-r+1}$. Recall also that $\operatorname{E}[X] = \sum_{x\in\mathbb{N}} P[X > x]$.
(4 Marks)

Exercise 4.4 A Bivariate Random Variable
Let (X, Y) be a continuous bivariate random variable
with density $f_{XY}: S \to \mathbb{R}^2$ given by

$$f_{XY}(x,y) = \begin{cases} c \cdot (x-y) & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le x, \\ 0 & \text{otherwise,} \end{cases}$$

where $c \in \mathbb{R}$ is a suitable constant.

i) Determine the constant c.
(2 Marks)

ii) Find $\mathrm{E}[X]$ and $\mathrm{E}[Y]$.
(2 Marks)

iii) Find $\mathrm{Var}[X]$ and $\mathrm{Var}[Y]$.

iv) Find the correlation coefficient ϱ_{XY}.
(2 Marks)

v) Find the density of $U = X/Y$.
(4 Marks)

Exercise 4.5 Minimum and Maximum of Two Independent Uniform Distributions
Let X_1 be a random variable following a continuous uniform distribution on $[0, 1]$ and X_2 an independent random variable following a continuous uniform distribution on $[-1/2, 1/2]$.

i) Let $Y = \min\{X_1, X_2\}$. Find the distribution of Y.
(2 Marks)

ii) Let $Y = \max\{X_1, X_2\}$. Find the distribution of Y.
(2 Marks)

Exercise 4.6 Sums of Independent, Continuous Random Variables
The *convolution* of two functions f and g is defined by

$$(f * g)(y) := \int_{-\infty}^{\infty} f(y-x)g(x)\,dx.$$

i) Let (X, f_X) and (Y, f_Y) be independent, continuous random variables. Show that their sum $Z = X + Y$ has density $f_Z = f_X * f_Y$.
(4 Marks)

ii) Suppose that the moment-generating functions m_X, m_Y and m_Z of X, Y and Z, respectively, exist. Show that $m_Z = m_X \cdot m_Y$.
(3 Marks)

iii) Let $X_1, \dots, X_n$ be a sample of size n from the distribution of a random variable X that follows a normal distribution with mean μ and variance σ^2. Find the distribution of

$$\overline{X} := \frac{1}{n}(X_1 + X_2 + \cdots + X_n).$$

(2 Marks)

iv) Based on the result in iii), verify directly that the weak law of large numbers holds for a sequence of i.i.d. normally distributed random variables.
(2 Marks)