Ve401 Probabilistic Methods in Engineering

Summer 2021 — Assigment 5

Date Due: 12:10 PM, Friday, the 18$^{\text{th}}$ of June 2021

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This assignment has a total of **(40 Marks)**.

Exercise 5.1 Reliability of Circuits

A system consists of two independent components. The life span (in hours) of the first component follows an exponential distribution with parameter β_1; the second has a lifespan in hours that follows the exponential distribution with parameter β_2.

i) Suppose the components are connected in series. Find the reliability function and the failure density of the system at time t.
(2 Marks)

ii) Suppose the components are connected in parallel. Find the reliability function and the failure density of the system at time t.
(2 Marks)

Exercise 5.2 Maximum Likelihood Estimation

The density of the *Laplace distribution* with parameter $\sigma > 0$ is given by

$$f(x) = \frac{1}{2\sigma} e^{-|x|/\sigma}, \qquad \sigma > 0,$$

i) Find the maximum-likelihood estimator $\hat{\sigma}$ for σ.
(3 Marks)

ii) Find the bias of $\hat{\sigma}$.
(1 Mark)

iii) Find the variance of $\hat{\sigma}$. What is the mean square error of $\hat{\sigma}$?
(3 Marks)

Exercise 5.3 Method-of-Moments Estimators

Let X be a continuous random variable with density

$$f_\theta(x) = \begin{cases} \dfrac{x}{\theta} e^{-x^2/(2\theta)} & \text{for } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is a parameter.

i) Calculate $\mathrm{E}[X]$ to find a method-of-moments estimator for the parameter θ.
(3 Marks)

ii) Calculate $\mathrm{E}[X^2]$ to find a different method-of-moments estimator for the parameter θ.
(3 Marks)

iii) Which of the above estimators is unbiased? Prove your assertion!
(2 Marks)

Exercise 5.4 Maximum-Likelihood Estimators Are Not Always Best

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a uniform continuous random variable[1] on the interval $[0, \theta]$, $\theta > 0$, i.e., having the density

$$f(x) = \begin{cases} 1/\theta, & x \in [0, \theta], \\ 0 & \text{otherwise.} \end{cases}$$

[1]This exercise is adapted from Example 5 of the very readable discussion of maximum likelihood estimators in L. Le Cam, *Maximum Likelihood: An Introduction*, International Statistical Review / Revue Internationale de Statistique, Vol. 58, No. 2 (Aug., 1990), pp. 153-171, http://www.jstor.org/stable/1403464

i) Show that the method-of-moments estimator for θ is

$$\hat{\theta}_{\text{MOM}} = 2\overline{X}$$

and verify that its means square error is

$$\text{MSE}(\hat{\theta}_{\text{MOM}}) = \frac{\theta^2}{3n}$$

(2 Marks)

ii) Show that the maximum-likelihood estimator for θ is

$$\hat{\theta}_{\text{ML}} = \max_{1 \le k \le n} X_k.$$

(2 Marks)

iii) Find a formula for the cumulative distribution function $P[\hat{\theta}_{\text{ML}} \le x] = P[X_1 \le x, \dots, X_n \le x]$ and then differentiate to obtain the density

$$f_{\hat{\theta}_{\text{ML}}}(x) = \begin{cases} nx^{n-1}/\theta^n, & 0 < x < \theta, \\ 0 & \text{otherwise.} \end{cases}$$

(2 Marks)

iv) Find the bias and the variance of $\hat{\theta}_{\text{ML}}$. Is $\hat{\theta}_{\text{ML}}$ unbiased? How do the bias and the variance behave as $n \to \infty$? Verify that the mean square error is given by

$$\text{MSE}(\hat{\theta}_{\text{ML}}) = \frac{2\theta^2}{(n+2)(n+1)}.$$

Compare with the mean square error of $\hat{\theta}_{\text{MOM}}$.
(4 Marks)

v) Show that the estimator

$$\hat{\theta}^* := \frac{n+2}{n+1} \max_{1 \le k \le n} X_k$$

has a smaller mean square error than $\hat{\theta}_{\text{ML}}$ whenever $n \ge 1$. Conclude that the method of maximum likelihood does not always yield the "best" estimator.
(2 Marks)

Exercise 5.5 Data Visualization and Interpretation

Consider the following data:

9.62	9.90	4.25	11.51	13.45	12.47	18.26	4.98	12.46	13.62
14.74	4.08	11.17	9.11	9.58	17.20	8.42	15.46	9.91	12.59
11.73	1.20	15.12	13.08	17.32	13.84	20.76	11.95	12.47	13.02
19.05	13.41	16.91	13.2	13.13	11.78	14.23	19.10	1.70	14.27
13.62	12.19	16.57	0.83	9.73	16.59	15.78	12.26	13.93	10.94
15.15	10.25	14.26	13.79	15.82	11.11	15.00	12.85	11.40	13.36
17.54	9.78	15.05	9.16	15.97	4.46	14.27	20.18	11.91	13.18
10.78	15.00	11.73	9.16	12.20	13.76	14.71	0.25	8.01	16.93
12.56	9.81	11.53	10.19	17.30	13.41	14.87	13.21	12.95	10.65
9.25	8.89	14.89	18.84	13.45	11.95	10.95	18.67	4.36	10.22

i) Find the quartiles and the interquartile range for the data.
(1 Mark)

ii) Create a histogram using the Freedman-Diaconis bin widths.
(2 Marks)

iii) Create a stem-and-leaf diagram for the data.
(2 Marks)

iv) Create a box-and-whisker diagram.
(2 Marks)

v) Does the data appear to come from a normal distribution? Give your reasoning!
(2 Marks)