Ve401 Probabilistic Methods in Engineering

Summer 2021 — Assigment 7

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JOINT INSTITUTE
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Date Due: 12:10 PM, Wednesday, the 7^{th} of July 2021

This assignment has a total of **(41 Marks)**.

Exercise 7.1 Fisher Test

Cloud seeding has been studied for many decades as a weather modification procedure.[a] It is claimed that mean rainfall from seeded clouds exceeds 25 acre-feet.

The rainfall in acre-feet from 20 clouds that were selected at random and seeded with silver nitrate follows:

18.0, 30.7, 19.8, 27.1, 22.3, 18.8, 31.8, 23.4, 21.2, 27.9,
31.9, 27.1, 25.0, 24.7, 26.9, 21.8, 29.2, 34.8, 26.7, 31.6.

Can you support the claim that mean rainfall from seeded clouds exceeds 25 acre-feet? What is the P-value of the test?
(3 Marks)

R. Munroe, *Correlation*, https://xkcd.com/892/

[a]For an interesting study of this subject, see the article in *Technometrics*, "A Bayesian Analysis of a Multiplicative Treatment Effect in Weather Modification", Vol. 17, pp. 161–166.

Exercise 7.2 Formalizing a Test procedure to a Fisher Test

A technician[1] wishes to buy a packet of 100 light bulbs. To ensure that fewer than 10 light bulbs are defective, she takes out 10 light bulbs randomly and tests them. If (and only if) none are defective, she will buy the packet.

Describe the actions of the technician in terms of a Fisher test. If she ends up buying the packet, what was the P-value of the test?
(4 Marks)

Exercise 7.3 Analysis of a Fisher Test

A manufacturer claims that the mean breaking strength of a certain fiber is more than $\mu = 150\,\text{psi}$. The breaking strength may be assumed to follow a normal distribution with $\sigma = 5\,\text{psi}$.

To provide evidence for this claim, a Fisher test is conducted with the null hypothesis $H_0\colon \mu \le 150\,\text{psi}$. If the test yields a P-value of less than 0.01, the manufacturers claim will be considered proven.

i) Suppose $n = 20$ samples are taken. How large does $\overline{x}$ need to be to achieve a P-value of 0.01 or less?
(2 Marks)

ii) Given $n = 20$ samples, how large would μ need to be so that this P-value is 99% certain to be achieved?
(2 Marks)

iii) Given $\mu = 155\,\text{psi}$, how large would n need to be so that this P-value is 99% certain to be achieved?
(2 Marks)

Exercise 7.4 Neyman-Pearson Decision Test

Medical researchers[2] have developed a new artificial heart constructed primarily of titanium and plastic. The heart will last and operate almost indefinitely once it is implanted in the patients body, but the battery pack needs to be recharged about every four hours.

It is claimed that the mean battery life μ exceeds 4 hours and a researcher would like to test the hypotheses

$$H_0\colon \mu \le 4\,\text{hours} \qquad\qquad H_1\colon \mu \ge 4.5\,\text{hours}$$

A random sample of 50 battery packs is selected and subjected to a life test. Assume that battery life is normally distributed with standard deviation $\sigma = 0.2$ hours.

[1]Adapted from U. Krengel, *Einführung in die Wahrscheinlichkeitstheorie und Statistik* [Introduction to Probability and Statistics], 5^{th} Ed., Vieweg Verlag, 2000

[2]Adapted from D.C. Montgomery and G.C. Runger, *Applied Statistics and Probability for Engineers*, 5^{th} Ed., John Wiley & Sons, 2011

i) Using the sample mean life span $\overline{X}$ as a test statistic, what is the critical region if $\alpha = 5\%$ is desired?
(2 Marks)

ii) Find the power of the test, i.e., the probability of rejecting H_0 if H_1 is true.
(1 Mark)

iii) What sample size would be required to obtain a power of at least 0.97?
(1 Mark)

iv) The sample mean life span turns out to be $\overline{x} = 4.05$ hours. Is H_0 rejected? Find a confidence interval for μ.
(2 Marks)

Exercise 7.5 A Noisy Bit

Consider a computer data storage medium such as a hard drive. When reading a memory bit b from the medium, the value of the bit may be either 0 or 1. However, the read process is affected by random noise N, independent of the value of B. Hence the value being read is

$$X = b + N,$$

where N is a random variable following a standard normal distribution.

When performing a read process, we test the hypotheses

$$H_0\colon b = 0, \qquad\qquad H_1\colon b = 1.$$

i) Given n, find the critical region for a Neyman-Pearson test with $\alpha = 0.01$.
(2 Marks)

ii) Find the required sample size (number of reads) to ensure that $\beta = 0.01$.
(2 Marks)

iii) Find the critical region in terms of $\overline{X}$ for the values of n and α given above.
(1 Mark)

Exercise 7.6 More on the Mean and Median

In this exercise, we will consider a continuous random variable X with density f_X, mean μ_X, median M_X and standard deviation σ_X.

i) Show that $g(y) := \mathrm{E}[|X - y|]$ is minimized by $y = M_X$.
(2 Marks)

ii) By exploiting $\mathrm{Var}[Y] \geq 0$ for any random variable Y, show that $\mathrm{E}[\sqrt{(X-\mu_X)^2}] \leq \sqrt{\mathrm{E}[(X-\mu_X)^2]}$.
(1 Mark)

iii) Show that

$$|\mu_X - M_X| \leq \sigma_X. \qquad (*)$$

(3 Marks)

iv) Let X follow an exponential distribution with parameter $\beta > 0$. Find M_X and verify $(*)$.
(2 Marks)

v) Suppose that X is symmetric about $a \in \mathbb{R}$. Show that $\mu_X = M_X$.
(2 Marks)

vi) Give an example of a continuous random variable for which the mean μ does not exist, but the median M does.
(1 Mark)

Exercise 7.7 Non-Parametric Tests on the Exponential Distribution
The following data was obtained from an exponential distribution with parameter $\beta > 0$:

1248.70	3658.24	1370.29	520.96	326.60	398.19	1649.17	1501.02	134.11	1460.08
456.31	4347.74	568.75	3.27	649.50	159.32	2201.73	259.67	2153.68	48.08
895.06	1742.06	1309.78	596.04	2083.94	1252.47	382.60	572.04	66.28	1730.83
181.10	501.53	45.27	642.19	1327.80	1904.87	263.87	397.67	1845.4	2206.01
255.09	192.48	1055.57	350.05	1708.98	476.20	2305.46	1112.15	381.43	1173.22

Perform the following tests of the hypothesis $H_0\colon \beta = 1/1000$:

i) A test based on the mean as in Exercise 6.3, vi).

ii) A sign test for the median.

iii) A Wilcoxon signed-rank test for the median.

Give the P-value of each test and interpret your results. What is your conclusion? Are all the above tests appropriate? Why or why not?
(6 Marks)